

## CANDY-PASSING GAMES ON GENERAL GRAPHS, II

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ABSTRACT. We give a new proof that any candy-passing game on a graph  $G$  with at least  $4|E(G)| - |V(G)|$  candies stabilizes. Unlike the prior literature on candy-passing games, we use methods from the general theory of chip-firing games which allow us to obtain a polynomial bound on the number of rounds before stabilization.

## 1. INTRODUCTION

We let  $G$  be an undirected graph and respectively denote the vertex and edge sets of  $G$  by  $V(G)$  and  $E(G)$ . The *candy-passing game on  $G$*  is defined by the following rules:

- At the beginning of the game,  $c > 0$  candies are distributed among  $|V(G)|$  students, each of whom is seated at some distinct vertex  $v \in V(G)$ .
- A whistle is sounded at a regular interval.
- Each time the whistle is sounded, every student who is able to do so passes one candy to each of his neighbors. (If at the beginning of this step a student holds fewer candies than he has neighbors, he does nothing.)

Tanton [6] introduced this game for cyclic  $G$ . The authors [4] extended the game to general graphs  $G$ .

The candy-passing game on  $G$  is a special case of the well-known *chip-firing game* on  $G$  introduced by Björner, Lovász, and Shor [2]. Furthermore, terminating candy-passing games on  $G$  are actually equivalent to terminating chip-firing games on  $G$ , by the following key theorem:

**Theorem 1** ([2]). *The initial configuration of a chip-firing game on  $G$  determines whether the game will terminate. If the game does terminate, then both the final configuration and length of the game are dependent only on the initial configuration.*

Terminating chip-firing games have been studied extensively and are surprisingly well-behaved. In addition to Theorem 1, it is known that terminating chip-firing processes finish in polynomial time (see [7]). Chip-firing games also have important applications; notably, they are related to Tutte polynomials (see [5]) and the critical groups of graphs (see [1]).

Infinite chip-firing games have received less attention, as the notion of an “end state” of such a game is ambiguous. By contrast, an infinite candy-passing game admits a clear stabilization condition: the game is said to have *stabilized* if the configuration of candy will never again change.

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2000 *Mathematics Subject Classification.* 05C35, 05C85, 68Q25 (Primary); 37B15, 68R10, 68Q80 (Secondary).

*Key words and phrases.* candy-passing, chip-firing, graph game, stabilization, polynomial time.

The second author gratefully acknowledges the support of a Harvard Mathematics Department Highbridge Fellowship.

The first author [3] studied the end behavior of candy-passing games on  $n$ -cycles, proving the eventual stabilization of any candy-passing game on an  $n$ -cycle with at least  $3n - 2$  candies. The authors [4] extended this analysis to arbitrary connected graphs  $G$ , showing that any candy-passing game on such  $G$  with at least  $4|E(G)| - |V(G)|$  candies will stabilize.

Here, we give a new proof of the stabilization result for general connected graphs, using methods which allow us to obtain a polynomial bound on the stabilization time. Our approach draws from the literature on chip-firing, using in particular a key result from Tardos's [7] proof that terminating chip-firing games conclude in polynomial time.

## 2. THE SETTING

As in the earlier work on candy-passing games, we refer to the interval between soundings of the whistle as a *round* of candy-passing. We denote by  $\varphi_t(v)$  the total number times a vertex  $v \in V(G)$  has passed candy by the end of round  $t$ .

Since infinite candy-passing games differ from infinite chip-firing games, we will continue to distinguish between “candies” and “chips.” However, we drop the student metaphor, treating the candy piles as belonging to the vertices of the graph  $G$ . For consistency, we denote the total number of candies in a candy-passing game by  $c$  throughout.

Abusing terminology slightly, we say that a vertex has *stabilized* in some round if, after that round, the amount of candy held by that vertex will not change during the remainder of the game.

For a vertex  $v \in V(G)$ , we denote the degree of  $v$  by  $\deg(v)$ . We say that a vertex  $v \in V(G)$  is *abundant* if it holds at least  $2\deg(v)$  pieces of candy.

Any vertex  $v \in V(G)$  with  $k \geq \deg(v)$  candies at the beginning of a round passes  $\deg(v)$  pieces of candy to its neighbors and can, at most, receive one piece of candy from each of its  $\deg(v)$  neighbors. Thus, such a vertex cannot end the round with more than  $k$  candies. In particular, then, the set of abundant vertices of  $G$  can only shrink over the course of a candy-passing game on  $G$ .

## 3. MAIN THEOREM

We will prove the following stabilization theorem:

**Theorem 2.** *Let  $G$  be a connected graph with diameter  $d$ . In any candy-passing game on  $G$  with*

$$c \geq 4|E(G)| - |V(G)|$$

*candies, every vertex  $v \in V(G)$  will stabilize within  $|V(G)| \cdot d \cdot c$  rounds.*

The stabilization component of Theorem 2 was obtained in [4, Theorem 2]. Our methods are inspired by those of Tardos [7]; they are essentially independent of the arguments used in [3] and [4].

We use the following lemma, which is a special case of Tardos's [7] Lemma 5:

**Lemma 3.** *Let  $v, v' \in V(G)$  be adjacent vertices of  $G$ . Then,  $|\varphi_t(v) - \varphi_t(v')| \leq c$  for all  $t$ .*

Additionally, we need an observation about the condition  $c \geq 4|E(G)| - |V(G)|$ .

**Lemma 4.** *For  $G$  a graph and  $c \geq 4|E(G)| - |V(G)|$ , in any chip-firing game on  $G$  with  $c$  candies there is at least one vertex  $v_* \in V(G)$  which passes candy every round.*

*Proof.* It suffices to find a vertex  $v_* \in V(G)$  which passes candy every round  $t$  during which some vertex  $v \in V(G)$  holds fewer than  $2\deg(v) - 1$  candies.

As observed above, it is not possible for a vertex  $v \in V(G)$  which is not abundant at the beginning of round  $t$  to become abundant after round  $t$ . However, the condition

$$c \geq 4|E(G)| - |V(G)|$$

guarantees that whenever some  $v \in V(G)$  holds fewer than  $2\deg(v) - 1$  candies there is also at least one abundant vertex  $v' \in V(G)$ . The existence of some vertex  $v_* \in V(G)$  which is abundant in every round when some vertex  $v \in V(G)$  has fewer than  $2\deg(v) - 1$  candies then follows immediately.  $\square$

**Remark.** Lemma 4 is, in some sense, dual to Tardos's [7] Lemma 4 which shows that for any terminating chip-firing game on  $G$  there is a distinguished vertex  $v_* \in V(G)$  which never fires.

We may now proceed with the proof of our main result:

*Proof of Theorem 2.* By Lemma 4, there is some vertex  $v_* \in V(G)$  which passes candy every round. Denoting the rounds by  $t = 1, 2, \dots$ , we then have  $\varphi_t(v_*) = t$  for all rounds  $t$ . By Lemma 3, we then know that

$$|\varphi_t(v_*) - \varphi_t(v)| \leq d \cdot c$$

for all  $t$  and  $v \in V(G)$ . Since  $\varphi_t(v_*)$  is strictly increasing in  $t$ , no  $v \in V(G)$  may fail to pass candy for more than  $d \cdot c$  rounds. In the worst case, all but one vertex pass candy in each round when some vertex does not pass candy; hence after  $|V(G)| \cdot d \cdot c$  rounds all the vertices of  $G$  pass candy every round.  $\square$

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